

ON UNIQUENESS OF DISTRIBUTION OF A RANDOM VARIABLE WHOSE INDEPENDENT COPIES SPAN A SUBSPACE IN L_p

S. ASTASHKIN, F. SUKOCHEV, AND D. ZANIN

ABSTRACT. Let $1 \leq p < 2$ and let $L_p = L_p[0, 1]$ be the classical L_p -space of all (classes of) p -integrable functions on $[0, 1]$. It is known that a sequence of independent copies of a mean zero random variable $f \in L_p$ spans in L_p a subspace isomorphic to some Orlicz sequence space l_M . We present precise connections between M and f and establish conditions under which the distribution of a random variable $f \in L_p$ whose independent copies span l_M in L_p is essentially unique.

1. INTRODUCTION

It is well known that the class of all subspaces of $L_1 = L_1(0, 1)$ is very rich and still does not have any reasonable description. If we consider only symmetric subspaces of L_1 , that is, subspaces with a symmetric basis or isomorphs of some symmetric function spaces, then these subspaces are known to be isomorphic to averages of Orlicz spaces [6, 13]. Far more information is available on subspaces of L_1 isomorphic to Orlicz spaces. First of all, an isomorph of an Orlicz sequence space $l_M \neq l_1$ in L_1 can always be given by the span of a sequence of independent identically distributed (i.i.d) random variables. The latter fact was discovered by M.I. Kadec in 1958 [8], who proved that for arbitrary $1 \leq p < q < 2$ there exists a symmetrically distributed function $f \in L_p$ (a q -stable random variable) such that the sequence $\{f_k\}_{k=1}^\infty$ of independent copies of f spans in L_p a subspace isomorphic to l_q .

This direction of study was taken further by J. Bretagnolle and D. Dacunha-Castelle (see [4, 5, 6]). In particular, D. Dacunha-Castelle showed that for every given mean zero $f \in L_p = L_p(0, 1)$, the sequence $\{f_k\}_{k=1}^\infty$ of its independent copies is equivalent in L_p to the unit vector basis of some Orlicz sequence space l_M [6, Theorem 1, p.X.8]. Moreover, J. Bretagnolle and D. Dacunha-Castelle proved that an Orlicz function space $L_M = L_M[0, 1]$ can be isomorphically embedded into the space L_p , $1 \leq p < 2$, if and only if M is equivalent to a p -convex and 2-concave Orlicz function on $[0, \infty)$ [5, Theorem IV.3]. Later on some of these results were independently rediscovered by M. Braverman [2, 3].

Note that the methods used in [4, 5, 6, 2, 3] depend heavily on the techniques related to the theory of random processes. In a recent paper [1], two first named co-authors suggested a different approach to study of this problem, which is based

2010 *Mathematics Subject Classification*: 46E30, 46B20, 46B09

Key words and phrases: L_p -space, Orlicz sequence space, independent random variables, p -convex function, q -concave function, subspaces

Authors acknowledge support from the ARC.

on methods and ideas from the interpolation theory of operators. In addition, it should be pointed out that papers [4, 5, 6, 2, 3] concern only with the verification of existence of a function f such that the sequence of its independent copies is equivalent in L_p to the unit vector basis in some Orlicz sequence space and do not address the question concerning the determination of f , whereas [1] is mainly focused on revealing precise connections between the Orlicz function and the distribution of corresponding random variable f . Among other results, in [1], it is shown the following. Let $1 \leq p < 2$ and let M be a p -convex and 2-concave Orlicz function on $[0, \infty)$ such that $M(t) \not\sim t^p$ for small $t > 0$ and the function

$$S(u) := -2pM(u) + (p+1)uM'(u) - u^2M''(u)$$

is positive on $(0, \infty)$, increasing and bounded on $(0, 1)$. Then, under some technical conditions on M (see [1, Proposition 12 and Theorem 15]) the unit vector basis in l_M is equivalent in L_p to the sequence $\{f_k\}_{k=1}^\infty$ of independent copies of an arbitrary mean zero function $f \in L_p$ such that its distribution function

$$n_f(\tau) := \lambda\{u : |f(u)| > \tau\}, \quad \tau > 0$$

(λ is the Lebesgue measure) is equivalent to the function $S(1/\tau)$ for $\tau \geq 1$.

The present paper continues this direction of research. Our main result (Theorem 1) is a somewhat surprising fact that in the case, when an Orlicz function M is ‘far’ from the extreme functions t^p and t^2 , $1 \leq p < 2$, the distribution of a random variable $f \in L_p$ whose independent copies span l_M essentially is equivalent to that of the function

$$\mathbf{m}(t) = \frac{1}{M^{-1}(t)}, \quad t > 0.$$

Theorem 1. *Let $1 \leq p < 2$ and let M be a p -convex and 2-concave Orlicz function. The following conditions are equivalent:*

- (i) *The function M is $(p + \varepsilon)$ -convex and $(2 - \varepsilon)$ -concave for some $\varepsilon > 0$;*
- (ii) *If a sequence $\{f_k\}_{k=1}^\infty$ of independent copies of a mean zero random variable $f \in L_p$ is equivalent in L_p to the unit vector basis $\{e_k\}_{k=1}^\infty$ in l_M , then the distribution function $n_f(\tau)$ is equivalent to that of \mathbf{m} for large τ .*
- (iii) *The function $\mathbf{m} \in L_p$ and any sequence of independent copies of a mean zero random variable equimeasurable with \mathbf{m} is equivalent in L_p to the unit vector basis in l_M .*

Observe that even in the simplest case, when $1 \leq p < q < 2$ and $M(t) = t^q$, $t \geq 0$, the theorem above complements the above-mentioned classical Kadec result [8], by establishing the uniqueness of the distribution of a mean zero random variable f whose independent copies span l_q in L_p .

It is worth noting that the assertion of Theorem 1 is in a sense sharp. Namely, in Proposition 13 we show that there exist two random variables x and y with non-equivalent distribution for large τ whose independent copies span in L_1 the same Orlicz space l_M , where M is equivalent to the function $t/\log(e/t)$ for small $t > 0$.

Note that in the special case $p = 1$, another attempt to describe the connection between the distribution of a random variable $f \in L_p$ and the corresponding Orlicz function M can be found in [12]. However, the methods used in [12] have a strong combinatorial flavor and formulas obtained there seem to be less accessible. Moreover, in [12] the question of uniqueness of distribution of f is not raised at all.

The proof of Theorem 1 is presented in Section 4. Two important components of the proof are Proposition 6 and Theorem 9, which are given in Sections 2 and 3, respectively.

We propose the following conjecture.

Conjecture 2. *Let $1 \leq p < 2$ and let M be a p -convex and 2-concave Orlicz function. If there is a unique (up to equivalence near 0) mean zero function f whose independent copies are equivalent in L_p to the unit vector basis in l_M , then M is $(p + \varepsilon)$ -convex and $(2 - \varepsilon)$ -concave for some $\varepsilon > 0$.*

2. PRELIMINARIES AND AUXILIARY RESULTS

2.1. Orlicz functions and spaces. For the theory of Orlicz spaces we refer to [9, 11].

Let M be an Orlicz function, that is, an increasing convex function on $[0, \infty)$ such that $M(0) = 0$. To any Orlicz function M we associate the Orlicz sequence space l_M of all sequences of scalars $a = (a_n)_{n=1}^\infty$ such that

$$\sum_{n=1}^{\infty} M\left(\frac{|a_n|}{\rho}\right) < \infty$$

for some $\rho > 0$. When equipped with the norm

$$\|a\|_{l_M} := \inf \left\{ \rho > 0 : \sum_{n=1}^{\infty} M\left(\frac{|a_n|}{\rho}\right) \leq 1 \right\},$$

l_M is a Banach space. Clearly, if $M(t) = t^p$, $p \geq 1$, then the Orlicz space l_M is the familiar space l_p . Moreover, the sequence $\{e_n\}_{n=1}^\infty$ given by

$$e_n = (\underbrace{0, \dots, 0}_{n-1 \text{ times}}, 1, 0, \dots)$$

is a Schauder basis in every Orlicz space l_M provided that M satisfies the Δ_2 -condition at zero, i.e., there are $u_0 > 0$ and $C > 0$ such that $M(2u) \leq CM(u)$ for all $0 < u < u_0$.

Similarly, if M is an Orlicz function, then the Orlicz function space $L_M = L_M[0, 1]$ consists of all measurable functions x on $[0, 1]$ such that the norm

$$\|x\|_{L_M} = \inf \left\{ u > 0 : \int_0^1 M(|x(t)|/u) dt \leq 1 \right\}$$

is finite.

Let $1 \leq p < q < \infty$. Given an Orlicz function M , we say that M is p -convex if the map $t \mapsto M(t^{1/p})$ is convex, and is q -concave if the map $t \mapsto M(t^{1/q})$ is concave. Throughout this paper, we assume that $M(1) = 1$ and that $M : [0, \infty) \rightarrow [0, \infty)$ is a bijection.

Careful inspection of the proof of [1, Lemma 5] establishes the following two lemmas.

Lemma 3. *Let $1 \leq p < \infty$. An Orlicz function $M : [0, \infty) \rightarrow [0, \infty)$ satisfying Δ_2 -condition at 0 is equivalent to a p -convex Orlicz function on the segment $[0, 1]$*

if and only if there exists a constant $C > 0$ such that for all $0 < s < 1$ and all $0 < t \leq 1$ we have

$$M(st) \leq Cs^p M(t).$$

Lemma 4. *Let $1 < q < \infty$. An Orlicz function $M : [0, \infty) \rightarrow [0, \infty)$ is equivalent to a q -concave Orlicz function on the segment $[0, 1]$ if and only if there exists a constant $C > 0$ such that for all $0 < s < 1$ and all $0 < t \leq 1$ we have*

$$C^{-1}s^q M(t) \leq M(st).$$

In what follows, by f^* we will denote the non-increasing right-continuous rearrangement of a random variable f , that is,

$$f^*(s) := \inf\{t : n_f(t) \leq s\},$$

where n_f is the distribution function of the random variable f . One says that random variables f and g are equimeasurable if $f^*(t) = g^*(t)$, $0 < t \leq 1$ (equivalently, $n_f(\tau) = n_g(\tau)$, $\tau > 0$). Finally, given two positive functions (quasinorms) f and g are said to be equivalent (we write $f \sim g$) if there exists a positive finite constant C such that $C^{-1}f \leq g \leq Cf$. Sometimes, we say that these functions are equivalent for large (or small) values of the argument, meaning that the preceding inequalities hold only for its specified values.

2.2. A condition for independent copies of a mean zero f to be equivalent in L_p to the unit vector basis of l_M . For a fixed $f \in L_1(0, 1)$, every $k \in \mathbb{N}$, and $t > 0$ we set

$$\bar{f}_k(t) := \begin{cases} f(t - k + 1), & t \in [k - 1, k), \\ 0, & \text{otherwise.} \end{cases}$$

The following assertion is an immediate consequence of the famous Rosenthal inequality [14] (or, its more general version due to Johnson and Schechtman [7]). It establishes a connection between the behaviour in L_p of an arbitrary sequence $\{f_k\}_{k=1}^\infty$ of independent copies of a mean zero random variable $f \in L_p$ and that of corresponding sequence $\{\bar{f}_k\}_{k=1}^\infty$ in the Banach sum $(L_p + L_2)(0, \infty)$ of the Lebesgue spaces $L_p(0, \infty)$ and $L_2(0, \infty)$.

Lemma 5. *Let $1 \leq p \leq 2$. For every finitely supported $a = (a_k)_{k=1}^\infty$ and for a mean zero random variable $f \in L_p(0, 1)$ we have*

$$\left\| \sum_{k=1}^\infty a_k f_k \right\|_p \sim \left\| \sum_{k=1}^\infty a_k \bar{f}_k \right\|_{L_p + L_2}.$$

Lemma 5 allows us to investigate sequences of independent identically distributed mean zero random variables in $L_p = L_p(0, 1)$.

Proposition 6. *Let $1 \leq p \leq 2$ and let $f \in L_p$ be a mean zero random variable. Then, a sequence $\{f_k\}_{k=1}^\infty$ of independent copies of the random variable f is equivalent (in L_p) to the unit vector basis in l_M if and only if*

$$(1) \quad \frac{1}{M^{-1}(t)} \sim \left(\frac{1}{t} \int_0^t f^*(s)^p ds \right)^{1/p} + \left(\frac{1}{t} \int_t^1 f^*(s)^2 ds \right)^{1/2}, \quad 0 < t \leq 1.$$

Proof. At first, we assume that a sequence $\{f_k\}_{k=1}^\infty$ of independent copies of f is equivalent in L_p to the unit vector basis in l_M . Then, we have

$$\left\| \sum_{k=1}^n e_k \right\|_{l_M} \sim \left\| \sum_{k=1}^n f_k \right\|_p \stackrel{\text{Lemma 5}}{\sim} \left\| \sum_{k=1}^n \bar{f}_k \right\|_{L_p+L_2}.$$

Since $1 \leq p \leq 2$, it follows that

$$\|x\|_{L_p+L_2} \sim \left(\int_0^1 x^*(s)^p ds \right)^{1/p} + \left(\int_1^\infty x^*(s)^2 ds \right)^{1/2}.$$

Therefore, from the equalities

$$\left(\sum_{k=1}^n \bar{f}_k \right)^*(s) = f^*\left(\frac{s}{n}\right), \quad s > 0,$$

and

$$\left\| \sum_{k=1}^n e_k \right\|_{l_M} = \inf \left\{ \rho > 0 : nM\left(\frac{1}{\rho}\right) \leq 1 \right\} = \frac{1}{M^{-1}(1/n)}, \quad n \geq 1,$$

it follows that

$$\begin{aligned} \frac{1}{M^{-1}(1/n)} &\sim \left(\int_0^1 (f^*\left(\frac{s}{n}\right))^p ds \right)^{1/p} + \left(\int_1^n (f^*\left(\frac{s}{n}\right))^2 ds \right)^{1/2} = \\ &= \left(n \int_0^{1/n} (f^*(s))^p ds \right)^{1/p} + \left(n \int_{1/n}^1 (f^*(s))^2 ds \right)^{1/2}, \quad n \geq 1. \end{aligned}$$

Let $t \in (1/(n+1), 1/n)$ for some $n \geq 1$. We clearly have $M^{-1}(1/n) \sim M^{-1}(t)$ and

$$\begin{aligned} \left(n \int_0^{1/n} (f^*(s))^p ds \right)^{1/p} + \left(n \int_{1/n}^1 (f^*(s))^2 ds \right)^{1/2} &\sim \\ &\sim \left(\frac{1}{t} \int_0^t (f^*(s))^p ds \right)^{1/p} + \left(\frac{1}{t} \int_t^1 (f^*(s))^2 ds \right)^{1/2}. \end{aligned}$$

The assertion (1) follows immediately from the equivalences above.

Conversely, by [6, Theorem 1, p.X.8] (see also [1, Theorem 9]), for every given mean zero $f \in L_p(0, 1)$ the sequence $\{f_k\}_{k=1}^\infty$ of independent copies of f is equivalent in L_p to the unit vector basis in some Orlicz sequence space l_N . Arguing in the same way as in the first part of the proof, we conclude that

$$\frac{1}{N^{-1}(t)} \sim \left(\frac{1}{t} \int_0^t f^*(s)^p ds \right)^{1/p} + \left(\frac{1}{t} \int_t^1 f^*(s)^2 ds \right)^{1/2}, \quad t \in (0, 1).$$

Taken together with (1) the equivalence above yields that the Orlicz functions M and N are equivalent on the segment $[0, 1]$ and thus, $l_N = l_M$. This completes the proof. \square

3. WHEN DOES THE EQUIVALENCE (1) HOLD FOR THE FUNCTION $f = \mathbf{m}$?

The following proposition provides necessary and sufficient conditions for the function \mathbf{m}^p to be equivalent to its Cesaro transform.

Proposition 7. *Let $1 \leq p < \infty$ and let M be a p -convex Orlicz function satisfying Δ_2 -condition at 0. The following conditions are equivalent:*

- (i) *The function M is equivalent on the segment $[0, 1]$ to a $(p + \varepsilon)$ -convex Orlicz function for some $\varepsilon > 0$;*

(ii)

$$\frac{1}{t} \int_0^t \mathbf{m}^p(s) ds \leq \text{const} \cdot \mathbf{m}^p(t), \quad t \in (0, 1).$$

Proof. Let the function φ be defined by setting

$$\varphi(t) = t\mathbf{m}^p(t), \quad t \in (0, 1).$$

(i) \rightarrow (ii). It suffices to show that

$$(2) \quad \int_0^t \frac{\varphi(s) ds}{s} \leq \text{const} \cdot \varphi(t), \quad t \in (0, 1).$$

It follows directly from the definitions that, for all $s \in (0, 1)$,

$$\sup_{0 < t \leq 1} \frac{\varphi(st)}{\varphi(t)} = s \cdot \sup_{0 < t \leq 1} \left(\frac{(M^{-1}(t))^{p+\varepsilon}}{(M^{-1}(st))^{p+\varepsilon}} \right)^{\frac{p}{p+\varepsilon}}.$$

Since M is $(p + \varepsilon)$ -convex, the mapping

$$t \rightarrow (M^{-1}(t))^{p+\varepsilon}, \quad t \in (0, 1],$$

is concave. In particular, we have

$$\frac{(M^{-1}(t))^{p+\varepsilon}}{(M^{-1}(st))^{p+\varepsilon}} \leq s^{-1}, \quad 0 < s, t \leq 1.$$

Therefore,

$$\sup_{t \in (0, 1)} \frac{\varphi(st)}{\varphi(t)} \leq s^{\frac{\varepsilon}{p+\varepsilon}}, \quad 0 < s \leq 1.$$

Applying now Lemma II.1.4 from [10], we infer (2) and this completes the proof of implication (i) \rightarrow (ii).

(ii) \rightarrow (i). Since M is p -convex, it follows that

$$\frac{M(s)}{s^p} \leq \frac{M(t)}{t^p}, \quad 0 \leq s \leq t \leq 1.$$

Replacing s with $M^{-1}(s)$ and t with $M^{-1}(t)$, we infer that φ is increasing.

By the assumption, we have

$$\int_0^t \frac{\varphi(s) ds}{s} \leq C\varphi(t), \quad t \in (0, 1),$$

for some $C > 0$. Take $s_0 < e^{-2C}$. We claim that

$$(3) \quad \sup_{t \in (0, 1)} \frac{\varphi(s_0 t)}{\varphi(t)} < 1.$$

Indeed, suppose that supremum in (3) equals 1. In particular, there exists $t \in (0, 1)$ such that $\varphi(s_0 t) > \varphi(t)/2$. Since φ is increasing and since $\log(s_0^{-1}) > 2C$, it follows that

$$\int_0^t \frac{\varphi(s) ds}{s} \geq \int_{s_0 t}^t \frac{\varphi(s) ds}{s} \geq \varphi(s_0 t) \log\left(\frac{t}{s_0 t}\right) > C\varphi(t).$$

This contradiction proves the claim.

According to (3), we can fix $a \in (0, 1)$ such that

$$(4) \quad \varphi(s_0 t) \leq a\varphi(t), \quad t \in (0, 1).$$

Without loss of generality, we can assume $a > s_0^{\frac{1}{1+p}}$. Hence, there exists $\varepsilon \in (0, 1)$ such that $a = s_0^{\frac{\varepsilon}{p+\varepsilon}}$.

For an arbitrary $s \in (0, 1]$ there exists $n \in \mathbb{N}$ such that $s \in (s_0^{n+1}, s_0^n)$. Since φ is increasing, it follows that

$$\varphi(st) \leq \varphi(s_0^n t) \stackrel{(4)}{\leq} s_0^{\frac{n\varepsilon}{p+\varepsilon}} \varphi(t) \leq s_0^{-\frac{\varepsilon}{p+\varepsilon}} s^{\frac{\varepsilon}{p+\varepsilon}} \varphi(t), \quad t \in (0, 1).$$

Hence, we have

$$\varphi(st) \leq \text{const} \cdot s^{\frac{\varepsilon}{p+\varepsilon}} \varphi(t), \quad s, t \in (0, 1)$$

or, equivalently,

$$(st)^{-\frac{\varepsilon}{p+\varepsilon}} \varphi(st) \leq \text{const} \cdot t^{-\frac{\varepsilon}{p+\varepsilon}} \varphi(t), \quad s, t \in (0, 1).$$

Therefore, it follows from the definition of φ that

$$M(st) \leq \text{const} \cdot s^{p+\varepsilon} \cdot M(t), \quad s, t \in (0, 1).$$

The argument is completed, by referring to Lemma 3. \square

Now, we prove a dual result.

Proposition 8. *Let M be a q -concave Orlicz function for some $1 < q < \infty$. The following conditions are equivalent:*

- (i) *The function M is equivalent to a $(q - \varepsilon)$ -concave Orlicz function for some $\varepsilon > 0$ on the segment $[0, 1]$;*
- (ii)

$$(5) \quad \frac{1}{t} \int_t^1 \mathfrak{m}^q(s) ds \leq \text{const} \cdot \mathfrak{m}^q(t), \quad t \in (0, 1).$$

Proof. Define the function ψ by setting

$$\psi(t) := t\mathfrak{m}^q(t), \quad t \in (0, 1).$$

(i) \rightarrow (ii). It suffices to verify that

$$\int_t^1 \frac{\psi(s) ds}{s} \leq \text{const} \cdot \psi(t), \quad t \in (0, 1).$$

We have

$$\sup \frac{\psi(st)}{\psi(t)} = s \cdot \sup \left(\frac{(M^{-1}(t))^{q-\varepsilon}}{(M^{-1}(st))^{q-\varepsilon}} \right)^{\frac{q}{q-\varepsilon}},$$

where the supremums are taken over all $t \in (0, 1)$ and $s > 1$ such that $0 < st \leq 1$. Since M is $(q - \varepsilon)$ -concave, it follows that the mapping

$$t \rightarrow (M^{-1}(t))^{q-\varepsilon}, \quad t \in (0, 1),$$

is convex. In particular, we have

$$\frac{(M^{-1}(t))^{q-\varepsilon}}{(M^{-1}(st))^{q-\varepsilon}} \leq s^{-1}, \quad s > 1, \quad 0 < st \leq 1.$$

Therefore,

$$\sup \frac{\psi(st)}{\psi(t)} \leq s^{-\frac{\varepsilon}{q-\varepsilon}} < 1,$$

where again the supremum is taken over all $t \in (0, 1)$ and $s > 1$ such that $0 < st \leq 1$. Applying now Lemma II.1.5 in [10], we infer (5).

(ii) \rightarrow (i). Since M is q -concave, it follows that

$$\frac{M(s)}{s^q} \geq \frac{M(t)}{t^q}, \quad 0 \leq s \leq t \leq 1.$$

Replacing s with $M^{-1}(s)$ and t with $M^{-1}(t)$, we infer that ψ is decreasing.

By the assumption, we have

$$\int_t^1 \frac{\psi(s) ds}{s} \leq C\psi(t), \quad t \in (0, 1),$$

for some $C > 0$. Take $s_0 > e^{2C}$. We claim that

$$(6) \quad \sup_{t \in (0, s_0^{-1})} \frac{\psi(s_0 t)}{\psi(t)} < 1.$$

Indeed, suppose that supremum in (6) equals 1. In particular, there exists $t \in (0, s_0^{-1})$ such that $\psi(s_0 t) \geq \psi(t)/2$. Since ψ is decreasing, it follows that

$$\int_t^1 \frac{\psi(s) ds}{s} \geq \int_t^{s_0 t} \frac{\psi(s) ds}{s} \geq \psi(s_0 t) \log\left(\frac{s_0 t}{t}\right) > C\psi(t).$$

This contradiction proves the claim.

According to (6), we can fix $b \in (0, 1)$ such that

$$(7) \quad \psi(s_0 t) \leq b\psi(t), \quad t \in (0, s_0^{-1}).$$

Without loss of generality, $b > s_0^{-1}$. Hence, there exists $\varepsilon > 0$ such that $b = s_0^{-\frac{\varepsilon}{q-\varepsilon}}$.

Let $s > 1$ and $0 < t < s^{-1}$. We can find $n \in \mathbb{N}$ such that $s \in (s_0^n, s_0^{n+1})$. Again appealing to the fact that ψ is decreasing, we have

$$\psi(st) \leq \psi(s_0^n t) \stackrel{(7)}{\leq} s_0^{-\frac{n\varepsilon}{q-\varepsilon}} \psi(t) \leq s_0^{\frac{\varepsilon}{q-\varepsilon}} s^{-\frac{\varepsilon}{q-\varepsilon}} \psi(t).$$

It follows that

$$\psi(st) \leq \text{const} \cdot s^{-\frac{\varepsilon}{q-\varepsilon}} \psi(t), \quad s > 1, t \in (0, s^{-1})$$

or, equivalently,

$$s^{\frac{\varepsilon}{q-\varepsilon}} \psi(s) \leq \text{const} \cdot t^{\frac{\varepsilon}{q-\varepsilon}} \psi(t), \quad 0 \leq t \leq s \leq 1.$$

Therefore, from the definition of ψ , we have

$$\frac{s}{M^{-1}(s)^{q-\varepsilon}} \leq \text{const} \cdot \frac{t}{M^{-1}(t)^{q-\varepsilon}}, \quad 0 \leq t \leq s \leq 1.$$

or

$$\text{const} \cdot s^{q-\varepsilon} \cdot M(t) \leq M(st), \quad \forall t, s \in (0, 1].$$

Applying Lemma 4, we complete the proof. \square

The following theorem answers the question stated in the title of the present section.

Theorem 9. *Let $1 \leq p < 2$ and let M be a p -convex and 2-concave Orlicz function. The following conditions are equivalent:*

- (i) *Equivalence (1) holds for $f = \mathbf{m}$.*
- (ii) *M is $(p + \varepsilon)$ -convex and $(2 - \varepsilon)$ -concave for some $\varepsilon > 0$.*

Proof. (ii) \Rightarrow (i). If M is $(p + \varepsilon)$ -convex for some $\varepsilon > 0$, then it follows from Proposition 7 that

$$(8) \quad \left(\frac{1}{t} \int_0^t \mathbf{m}^p(s) ds \right)^{1/p} \leq \text{const} \cdot \mathbf{m}(t), \quad t \in (0, 1).$$

If M is $(2 - \varepsilon)$ -concave for some $\varepsilon > 0$, then Proposition 8 implies

$$(9) \quad \left(\frac{1}{t} \int_t^1 \mathbf{m}^2(s) ds \right)^{1/2} \leq \text{const} \cdot \mathbf{m}(t), \quad t \in (0, 1).$$

Observe now that the inequality

$$(10) \quad \mathbf{m}(t) \leq \left(\frac{1}{t} \int_0^t \mathbf{m}^p(s) ds \right)^{1/p}, \quad t \in (0, 1)$$

holds trivially, due to the fact that \mathbf{m} is decreasing. The equivalence (1) for $f = \mathbf{m}$ follows immediately from (8), (9) and (10).

(i) \Rightarrow (ii). Suppose that (1) holds for $f = \mathbf{m}$. Then, we have (8) and (9). Applying Propositions 7 and 8, we obtain that M is $(p + \varepsilon)$ -convex and $(2 - \varepsilon)$ -concave for some $\varepsilon > 0$, and the proof is completed. \square

4. WHEN DOES EQUIVALENCE (1) HOLD FOR A UNIQUE f (UP TO EQUIVALENCE NEAR 0)?

This section contains the proof of Theorem 1.

Proof of Theorem 1. The implication (ii) \rightarrow (iii) is obvious and the implication (iii) \rightarrow (i) follows by combining results of Proposition 6 and Theorem 9.

(i) \rightarrow (ii). We begin with the following technical lemma.

Lemma 10. *Let $1 \leq p < \infty$, $1 < q < \infty$ and let M be an Orlicz function.*

(i) *If M is $(q - \varepsilon)$ -concave for some $\varepsilon > 0$, then*

$$N \sup_{t>0} \frac{\mathbf{m}^q(Nt)}{\mathbf{m}^q(t)} \rightarrow 0, \quad N \rightarrow \infty.$$

(ii) *If M is $(p + \varepsilon)$ -convex for some $\varepsilon > 0$, then*

$$\frac{1}{N} \cdot \sup_{t>0} \frac{\mathbf{m}^p(\frac{t}{N})}{\mathbf{m}^p(t)} \rightarrow 0, \quad N \rightarrow \infty.$$

Proof. Proofs of (i) and (ii) are very similar. So, we prove (i) only.

Since M is $(q - \varepsilon)$ -concave, it follows that the mapping

$$t \rightarrow \frac{M(t)}{t^{q-\varepsilon}}, \quad t > 0,$$

is decreasing. Hence, the mapping

$$t \rightarrow t \mathbf{m}^{q-\varepsilon}(t) = \frac{t}{(M^{-1}(t))^{q-\varepsilon}}, \quad t > 0,$$

is also decreasing. Therefore,

$$N^{\frac{q}{q-\varepsilon}} \sup_{t>0} \frac{\mathbf{m}^q(Nt)}{\mathbf{m}^q(t)} = \left(\sup_{t>0} \frac{N t \mathbf{m}^{q-\varepsilon}(Nt)}{t \mathbf{m}^{q-\varepsilon}(t)} \right)^{\frac{q}{q-\varepsilon}} \leq 1,$$

whence

$$N \sup_{t>0} \frac{\mathfrak{m}^q(Nt)}{\mathfrak{m}^q(t)} \leq N^{-\frac{\varepsilon}{q-\varepsilon}} \rightarrow 0 \quad \text{if } N \rightarrow \infty.$$

□

Now, let M be a $(p+\varepsilon)$ -convex and $(2-\varepsilon)$ -concave Orlicz function and let f be a mean zero function from L_p . Suppose that the sequence $\{f_k\}_{k=1}^\infty$ of independent copies of f is equivalent to the unit vector basis $\{e_k\}_{k=1}^\infty$ in l_M . It suffices to show that the functions f^* and \mathfrak{m} are equivalent for small values of argument. For simplicity we abuse the notation assuming that $f = f^*$. By Proposition 6 we know that the equivalence (1) holds for f , that is,

$$(11) \quad \mathfrak{m}(t) \sim \left(\frac{1}{t} \int_0^t f(s)^p ds \right)^{1/p} + \left(\frac{1}{t} \int_t^1 f(s)^2 ds \right)^{1/2}, \quad t \in (0, 1).$$

By Theorem 9, we also have

$$(12) \quad \mathfrak{m}(t) \sim \left(\frac{1}{t} \int_0^t \mathfrak{m}(s)^p ds \right)^{1/p} + \left(\frac{1}{t} \int_t^1 \mathfrak{m}(s)^2 ds \right)^{1/2}, \quad t \in (0, 1).$$

Observe now that the estimate

$$(13) \quad f(t) \leq C_1 \cdot \mathfrak{m}(t), \quad t \in (0, 1),$$

for some $C_1 > 0$ follows immediately from (11) and the (already used) inequality

$$f(t) \leq \left(\frac{1}{t} \int_0^t f(s)^p ds \right)^{1/p}, \quad t \in (0, 1).$$

Thus, we need to show that the estimate

$$(14) \quad \mathfrak{m}(t) \leq \text{const} \cdot f(t), \quad t \in (0, 1),$$

holds for all sufficiently small $t \in (0, 1)$. By Propositions 7 and 8, there exists a constant $C_0 > 0$ such that

$$(15) \quad \frac{1}{t} \int_0^t \mathfrak{m}^p(s) ds \leq C_0^p \mathfrak{m}^p(t), \quad t \in (0, 1),$$

$$(16) \quad \frac{1}{t} \int_t^1 \mathfrak{m}^2(s) ds \leq C_0^2 \mathfrak{m}^2(t), \quad t \in (0, 1).$$

Moreover, there is a constant $C > 0$ such that for a given $t \in (0, 1)$, from (11) it follows that either

$$(17) \quad \left(\frac{1}{t} \int_t^1 f^2(s) ds \right)^{1/2} \geq \frac{1}{2C} \mathfrak{m}(t),$$

or

$$(18) \quad \left(\frac{1}{t} \int_0^t f^p(s) ds \right)^{1/p} \geq \frac{1}{2C} \mathfrak{m}(t).$$

By Lemma 10, we can fix N so large that

$$(19) \quad \sup_{t>0} \frac{\mathfrak{m}^2(Nt)}{\mathfrak{m}^2(t)} \leq \frac{1}{8NC^2C_1^2}, \quad \sup_{t>0} \frac{\mathfrak{m}^p(\frac{t}{N})}{\mathfrak{m}^p(t)} \leq \frac{N}{2^{p+1}C_1^pC^p}.$$

Let $t \in (0, 1/N)$. Firstly, we consider the situation when (17) holds. Taking squares in this inequality and then applying (13), we obtain

$$\begin{aligned} \frac{1}{4C^2} \mathbf{m}^2(t) &\leq \frac{1}{t} \int_t^1 f^2(s) ds = \frac{1}{t} \int_t^{Nt} f^2(s) ds + \frac{1}{t} \int_{Nt}^1 f^2(s) ds \\ &\leq (N-1)f^2(t) + \frac{NC_1^2}{Nt} \int_{Nt}^1 \mathbf{m}^2(s) ds. \end{aligned}$$

Hence, by (16), we have

$$\frac{1}{4C^2} \mathbf{m}^2(t) \leq (N-1)f^2(t) + NC_1^2 C_0^2 \mathbf{m}^2(Nt).$$

Combining the latter estimate with the first inequality in (19), we obtain

$$(N-1)f^2\left(\frac{t}{N}\right) \geq (N-1)f^2(t) \geq \frac{1}{4C^2} \mathbf{m}^2(t) - NC_1^2 C_0^2 \mathbf{m}^2(Nt) \stackrel{(19)}{\geq} \frac{1}{8C^2} \mathbf{m}^2(t).$$

If (18) holds, then

$$\frac{1}{2^p C^p} \mathbf{m}^p(t) \leq \frac{1}{t} \int_0^t f^p(s) ds = \frac{1}{t} \int_0^{t/N} f^p(s) ds + \frac{1}{t} \int_{t/N}^t f^p(s) ds.$$

Taking (13) and (15) into account, we obtain

$$\begin{aligned} \frac{1}{2^p C^p} \mathbf{m}^p(t) &\leq \frac{C_1^p/N}{t/N} \int_0^{t/N} \mathbf{m}^p(s) ds + \left(1 - \frac{1}{N}\right) f^p\left(\frac{t}{N}\right) \\ &\leq \frac{1}{N} C_1^p C_0^p \mathbf{m}^p\left(\frac{t}{N}\right) + \left(1 - \frac{1}{N}\right) f^p\left(\frac{t}{N}\right). \end{aligned}$$

We infer from this estimate and the second inequality in (19) that

$$\left(1 - \frac{1}{N}\right) f^p\left(\frac{t}{N}\right) \geq \frac{1}{2^p C^p} \mathbf{m}^p(t) - \frac{1}{N} C^p C_0^p \mathbf{m}^p\left(\frac{t}{N}\right) \stackrel{(19)}{\geq} \frac{1}{2^{p+1} C^p} \mathbf{m}^p(t).$$

In either case, we have

$$f\left(\frac{t}{N}\right) \geq \text{const} \cdot \mathbf{m}(t), \quad t \in (0, \frac{1}{N}),$$

for a universal constant. Since $\mathbf{m}(t) \sim \mathbf{m}(t/N)$, it follows that

$$f(t) \geq \text{const} \cdot \mathbf{m}(t), \quad t \in (0, \frac{1}{N^2}).$$

The latter inequality together with (13) suffices to conclude the proof of implication (i) \rightarrow (ii). \square

5. SHARPNESS OF THEOREM 1

Let $\{h_k\}_{k=1}^\infty$ (respectively, $\{g_k\}_{k=1}^\infty$) be a sequence of pairwise disjoint measurable subsets of $(0, 1)$ such that $\lambda(h_k) = 2^{-k-2^k}$ (respectively, $\lambda(g_k) = 4^{-k-4^k}$), $k \geq 1$. We define functions $x, y \in L_1(0, 1)$ by setting

$$(20) \quad x = \sum_{k=1}^\infty 2^{2^k} \chi_{h_k}, \quad y = \sum_{k=1}^\infty 4^{4^k} \chi_{g_k},$$

(χ_c is the indicator function of a set c).

Lemma 11. *We have*

$$\int_0^1 \min\{x(s), tx^2(s)\} ds \sim \int_0^1 \min\{y(s), ty^2(s)\} ds \sim \frac{1}{\log(e/t)}, \quad 0 < t \leq 1.$$

Proof. It is clear that

$$\int_0^1 \min\{x(s), tx^2(s)\} ds = \sum_{2^{2^k} \geq 1/t} 2^{2^k} \cdot 2^{-k-2^k} + t \cdot \sum_{2^{2^k} < 1/t} 2^{2^{k+1}} \cdot 2^{-k-2^k}.$$

Let $t < 1/4$. If m is the maximal positive integer such that $2^{2^m} < 1/t$, then

$$\int_0^1 \min\{x(s), tx^2(s)\} ds = \sum_{k=m+1}^{\infty} 2^{-k} + t \cdot \sum_{k=1}^m 2^{2^k-k} = 2^{-m} + t \cdot \sum_{k=1}^m 2^{2^k-k}.$$

Also, we have

$$\sum_{k=1}^m 2^{2^k-k} \leq 2^{2^m-m} + (m-1) \cdot 2^{2^{m-1}-m+1} \leq 2^{2^m-m} + 2^{2^{m-1}} \leq 2 \cdot 2^{2^m-m}.$$

Therefore, we obtain

$$2^{-m} \leq \int_0^1 \min\{x(s), tx^2(s)\} ds \leq 2^{-m} + 2t \cdot 2^{2^m-m} \leq 3 \cdot 2^{-m}.$$

It follows now from the definition of the number m that

$$\frac{1}{\log_2(1/t)} \leq \int_0^1 \min\{x(s), tx^2(s)\} ds \leq \frac{6}{\log_2(1/t)}.$$

The similar equivalence for y follows *mutatis mutandi*. □

Lemma 12. *Distributions of the functions x and y are not equivalent.*

Proof. Suppose that $n_x(Ct) \leq Cn_y(t)$, $t > 0$. Fix k such that

$$2^{2^{k+1}} > \log_2 C + 1$$

and select t such that both t and Ct belong to the interval $(2^{2^{k+1}}, 2^{2^{k+2}})$. Then, we have

$$n_x(Ct) = n_x(2^{2^{k+1}}) \geq 2^{-(2k+2)-2^{2k+2}}$$

and

$$n_y(t) = n_y(4^{4^k}) \leq 2 \cdot 4^{-(k+1)-4^{k+1}} = 2^{-2k-1-2^{2k+3}}.$$

It follows from the preceding inequalities that

$$2^{2k+2+2^{2k+2}} \geq \frac{1}{C} \cdot 2^{2k+1+2^{2k+3}}$$

or, equivalently,

$$2k+2+2^{2k+2} \geq -\log_2(C) + 2k+1+2^{2k+3}.$$

Clearly, the latter inequality contradicts to the choice of k . □

Let $\{x_k\}_{k=1}^{\infty}$ (respectively, $\{y_k\}_{k=1}^{\infty}$) be a sequence of independent copies of a mean zero random variable equimeasurable with x (respectively, y), where x and y are defined in (20). Let us show that the sequences $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ span in L_1 the same Orlicz space l_M , where M is equivalent to the function $t/\log(e/t)$ for small $t > 0$. Note that M does not satisfy condition (i) of Theorem 1; more

precisely, M is not $(1 + \varepsilon)$ -convex for any $\varepsilon > 0$. Taking into account Lemma 5, it suffices to prove the following proposition.

Proposition 13. *For every finitely supported $a = (a_k)_{k=1}^\infty$, we have*

$$\left\| \sum_{k=1}^n a_k \bar{x}_k \right\|_{L_1 + L_2} \sim \left\| \sum_{k=1}^n a_k \bar{y}_k \right\|_{L_1 + L_2} \sim \|(a_k)_{k=1}^\infty\|_{l_M}.$$

Proof. Define the Orlicz function N by setting

$$N(t) = \begin{cases} t^2, & t \in (0, 1) \\ 2t - 1, & t \geq 1. \end{cases}$$

It is easy to check that $\|z\|_{L_1 + L_2} \sim \|z\|_{L_N}$ for every $z \in L_1 + L_2$, where L_N is the function Orlicz space on $[0, 1]$.

Setting

$$M(t) = \int_0^1 N(tx(s)) ds, \quad t > 0,$$

we obtain

$$\begin{aligned} \left\| \sum_{k=1}^\infty a_k \bar{x}_k \right\|_{L_N} \leq 1 &\iff \int_0^\infty N\left(\sum_{k=1}^\infty |a_k| |\bar{x}_k(s)|\right) ds \leq 1 \\ &\iff \sum_{k=1}^\infty \int_0^1 N(|a_k| |x_k(s)|) ds \leq 1 \\ &\iff \sum_{k=1}^\infty M(a_k) \leq 1 \iff \|a\|_{l_M} \leq 1. \end{aligned}$$

Therefore,

$$\left\| \sum_{k=1}^\infty a_k \bar{x}_k \right\|_{L_1 + L_2} \sim \|a\|_{l_M}.$$

Since $N(t) \sim \min\{t, t^2\}$ ($t > 0$), it follows that

$$M(t) \sim \int_0^1 \min\{tx(s), (tx(s))^2\} ds,$$

and from Lemma 11 it follows that

$$M(t) \sim \frac{t}{\log(e/t)}, \quad 0 < t \leq 1.$$

This proves the assertion for the sequence $\{x_k\}$. The proof of the similar assertion for $\{y_k\}$ is the same. \square

REFERENCES

- [1] Astashkin S., Sukochev F. *Orlicz sequence spaces spanned by identically distributed independent random variables in L_p -spaces*. J. Math. Anal. Appl. **413** (2014), no. 1, 1–19.
- [2] Braverman M. Sh. *On some moment conditions for sums of independent random variables*, Probab. Math. Statist. **14** (1993), no. 1, 45–56.
- [3] Braverman M. Sh. *Independent random variables in Lorentz spaces*, Bull. London Math. Soc. **28** (1996), no. 1, 79–87.
- [4] Bretagnolle J., Dacunha-Castelle D. *Mesures aléatoires et espaces d'Orlicz*, (French) C. R. Acad. Sci. Paris Ser. A-B **264** (1967), A877–A880.

- [5] Bretagnolle J., Dacunha-Castelle D. *Application de l'étude de certaines formes linéaires aléatoires au plongement d'espaces de Banach dans des espaces L^p* , Ann. Sci. Ecole Norm. Sup. 2(1969), no. 5, 437-480.
- [6] Dacunha-Castelle D. *Variables aléatoires échangeables et espaces d'Orlicz*. Séminaire Maurey-Schwartz 1974-1975: *Espaces L^p , applications radonifiantes et géométrie des espaces de Banach*. Exp. Nos. X et XI, 21 pp. Centre Math., École Polytech., Paris, 1975.
- [7] Johnson W., Schechtman G. *Sums of independent random variables in rearrangement invariant function spaces*, Ann. Probab. **17** (1989), 789-808.
- [8] Kadec M. I. *Linear dimension of the spaces L_p and l_q* , Uspehi Mat. Nauk, **13**:6(84) (1958), 95-98. (in Russian)
- [9] Krasnoselskii M., Rutickii J. *Convex Functions and Orlicz Spaces*. Fizmatgiz, Moscow 1958 (in Russian); English transl.: Noordhoff, Groningen 1961.
- [10] Krein S., Petunin Ju., Semenov E. *Interpolation of linear operators*, Nauka, Moscow, 1978 (in Russian); English translation in Translations of Math. Monographs, Vol. **54**, Amer. Math. Soc., Providence, RI, 1982.
- [11] Lindenstrauss J., Tzafriri L. *Classical Banach Spaces II. Function spaces*. Berlin-Heidelberg-New York: Springer-Verlag, 1979.
- [12] Schütt C. *On the embedding of 2-concave Orlicz spaces into L_1* , Studia Math. **113** (1995), no. 1, 73-80.
- [13] Raynaud Y., Schütt C. *Some results on symmetric subspaces of L^1* , Stud. Math., **89**(1988), 27-35.
- [14] Rosenthal H.P. *On the subspaces of L_p ($p > 2$) spanned by sequences of independent random variables*, Isr. J. Math. **8** (1970), 273-303.

SAMARA STATE UNIVERSITY, PAVLOVA 1, SAMARA, 443011, RUSSIA

E-mail address: `astash@samsu.ru`

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW SOUTH WALES, SYDNEY, 2052, AUSTRALIA.

E-mail address: `f.sukochev@unsw.edu.au`

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW SOUTH WALES, SYDNEY, 2052, AUSTRALIA.

E-mail address: `d.zanin@unsw.edu.au`